

A NOTE ON OPERATOR SEMIGROUPS ASSOCIATED TO CHAOTIC FLOWS

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ABSTRACT. The transfer operator associated to a flow (continuous time dynamical system) is a one-parameter operator semigroup. We consider the operator-valued Laplace transform of this one-parameter semigroup. Estimates on the Laplace transform have been used in various settings in order to show the rate at which the flow mixes. Here we consider the case of exponential mixing or rapid mixing (super polynomial). We develop the operator theory framework amenable to this setting and show that the same estimates may be used to produce results, in terms of the operators, which go beyond the results for the rate of mixing. Such results are useful for obtaining other statistical properties of the dynamical system.

1. INTRODUCTION

Flows are important dynamical systems, arguable the origin of much of the research in the area of dynamical systems. It has proved significantly more difficult to study strong statistical properties of flows compared to corresponding questions for discrete time systems. Of particular importance is proving the rate of mixing of a given system or family of systems. Substantial initial progress was made by studying the Laplace transform of the correlation function [22, 8]. A certain estimate (the oscillatory cancellation estimate pioneered by Dolgopyat [8]) can then be translated into an exponential mixing estimate for the flow. These ideas were developed by Liverani to the closely related question of studying the resolvent operator of the infinitesimal generator of the semigroup of transfer operators [17]. An identical argument is used by Baladi and Liverani [2], and by Giuletti, Pollicott and Liverani [12]. We further develop these ideas, extending the idea of considering the operator-valued Laplace transform of the Transfer operator [3] and show that one may squeeze yet more information from this line of thinking. The case for exponentially mixing flows and rapid mixing flows are presented side-by-side in the same language and so are easily comparable.

The improved operator-theoretic result is of interest in several ways. Firstly that beyond the rate of mixing there are many other statistical properties which can often be deduced from the spectral results [14, §9] and cannot be deduced directly from the rate of mixing. A powerful use of the functional analysis is with *operator-renewal* techniques [23] which are useful for studying slowly mixing systems and infinite ergodic systems [20]. Another important use of the functional analysis is for studying how statistical properties behave under perturbations of the dynamical system [15]. Such perturbations could be deterministic or random. Moreover the same ideas (as interpreted in [16]) can even be used to study the physically important question of understanding coupled dynamical systems.

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The improvements are as follows. The calculation involved is completely streamlined. This makes it clear that the constant obtained in the decay rate (in terms of degree of differentiability of the observable) cannot be improved without additional ideas. We avoid the need for Liverani's "silly preliminary fact" [17, Lemma 2.14]. Additionally we are able to obtain a spectral decomposition (2.3) of the transfer operator in a sense similar, although weaker, than the results of Tsujii [24, 25, 26]. This means we obtain a precise description of the mixing and moreover it is to be expected that further information concerning other statistical properties can be obtained from this operator-theoretic representation. The result is in a form especially amenable to the ideas of [15] regarding the use of operator perturbation theory in order to understand various questions in dynamical systems.

Note that in this document we do not prove the required estimates for any particular dynamical systems with respect to any particular Banach space. Rather we isolate the abstract argument and make some improvements to this. It is an important question and a subject of ongoing research to investigate the rate of mixing (and other fine statistical properties) for a broad spectrum of flows. The method we are discussing (i.e., functional analysis applied to dynamical systems) requires as a first step the choice or design of a Banach space on which the one-parameter family of transfer operators acts "nicely". Moreover, at this point in time, to answer such questions for flows, no one knows a method which does not involve functional-analytic ideas to some extent. Designing appropriate Banach spaces and proving such estimates for systems of interest (including many physically relevant systems with discontinuities and singularities) remains an important subject of ongoing research [7, 4]. In many cases the appropriate choice of Banach space is far from obvious. In this note we are able to reduce the assumptions that such a Banach space must satisfy in order to be useful and consequently simplify the search for and construction of the dynamically relevant Banach spaces. In particular we avoid the requirement that the one-parameter semigroup is strongly continuous.

In view of potential numerical applications throughout the argument we will keep track of all the relevant constants. In Section 2 we present the results in two theorems, one concerning the exponentially mixing case and the other concerning the rapid mixing case. In Section 3 we give details of systems where the required assumptions have already been shown to be satisfied. We expect these assumptions will soon be shown to be satisfied in many more settings. Section 4 and Section 5 are devoted to the proofs of the results.

2. RESULTS

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be Banach spaces such that $\mathcal{A} \supset \mathcal{B}$ and $\|\cdot\|_{\mathcal{A}} \leq \|\cdot\|_{\mathcal{B}}$.¹ We consider a family of bounded linear operators $T_t : \mathcal{B} \rightarrow \mathcal{B}$ parameterised by $t \geq 0$ and such that

$$T_0 = \text{id}, \quad T_s \circ T_t = T_{s+t} \quad \text{for all } t, s \geq 0,$$

and that $\|T_t\|_{\mathcal{B}} \leq C_1$ for some $C_1 > 0$. In other words $T_t : \mathcal{B} \rightarrow \mathcal{B}$ is a bounded one-parameter semigroup.² Let $\mathcal{B}(\mathcal{B}, \mathcal{B})$ denote the Banach space of bounded linear operators $T : \mathcal{B} \rightarrow \mathcal{B}$ equipped with the standard operator norm which we denote $\|T\|_{\mathcal{B}}$. We also define a weaker operator norm

$$\|T\|_{\mathcal{B} \rightarrow \mathcal{A}} := \sup\{\|T\mu\|_{\mathcal{A}} : \mu \in \mathcal{B}, \|\mu\|_{\mathcal{B}} \leq 1\}. \quad (2.1)$$

¹In actual fact one needs only the Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ equipped with an auxiliary, weaker norm $\|\cdot\|_{\mathcal{A}}$. However in this case one can always define \mathcal{A} to be the completion of \mathcal{B} with respect to $\|\cdot\|_{\mathcal{A}}$ and so without loss of generality we give the assumptions as above.

²The boundedness requirement is essentially superfluous since any one-parameter semigroup satisfies a bound of the form $\|T_t\|_{\mathcal{B}} \leq Ce^{\gamma t}$ and in this case we may simply consider the operator $\tilde{T}_t := e^{-\gamma t}T_t$ and proceed as before.

It would be unrealistic in the intended applications to hope that the semigroup is norm continuous. We merely require the following, substantially weaker, continuity condition.

Assumption 1 (Weak-Lipschitz). *There exists $C_2 > 0$ such that*

$$\frac{1}{t} \|T_t - \text{id}\|_{\mathcal{B} \rightarrow \mathcal{A}} \leq C_2 \quad \text{for all } t \geq 0.$$

See Section 3 for discussion of this assumption and how it is natural in the intended applications. For all $z \in \mathbb{C}$, $\Re(z) > 0$ let $R(z) \in \mathcal{B}(\mathcal{B}, \mathcal{B})$ be defined by the Bochner integral

$$R(z) := \int_0^\infty e^{-zt} T_t \, dt. \quad (2.2)$$

Since the semigroup is bounded we know that $\|R(z)\|_{\mathcal{B}} \leq C_1 \Re(z)^{-1}$ for all $\Re(z) > 0$ but we need a bit more information concerning $R(z)$.

Assumption 2 (Lasota-Yorke). *There exists $\lambda > 0$ such that the essential spectral radius of $R(z) : \mathcal{B} \rightarrow \mathcal{B}$ is not greater than $(\Re(z) + \lambda)^{-1}$ for all $\Re(z) > 0$.*

In all cases we will assume the both Assumption 1 and Assumption 2 hold. In addition we will assume that one of the two following assumptions holds. The first is an oscillatory cancellation type estimate of the form used by Dolgopyat in the study of Anosov flows [8].

Assumption 3A (Exponential). *There exists $\beta, \alpha, C_3 > 0$ and $\gamma \in (0, 1/\ln(1 + \lambda/\alpha))$ such that, for all $\Re(z) = \alpha$, $|\Im(z)| \geq \beta$,*

$$\|R(z)^{\tilde{n}}\|_{\mathcal{B}} \leq C_3 (\Re(z) + \lambda)^{-\tilde{n}}, \quad \text{where } \tilde{n} = \lceil \gamma \ln |\Im(z)| \rceil.$$

An alternative and far weaker assumption is the following estimate of the form used by Dolgopyat in the study of the prevalence of rapid mixing among Axiom A flows [9].

Assumption 3B (Rapid). *There exists $\beta, C_4, s, r > 0$ such that $R(z)$ admits a holomorphic extension to the set $\{z \in \mathbb{C} : |\Im(z)| \geq \beta, \Re(z) \geq -|\Im(z)|^{-r}\}$ and on this set*

$$\|R(z)\|_{\mathcal{B}} \leq C_4 |\Im(z)|^s.$$

That Assumption 3A is stronger than Assumption 3B can be seen from the calculations in Section 4.

The *generator* of the one-parameter semigroup T_t is the linear operator defined by

$$Z\mu := \lim_{t \rightarrow 0} \frac{1}{t} (T_t \mu - \mu)$$

the domain of Z , which we denote $\text{Dom}(Z)$, being the set of $\mu \in \mathcal{B}$ for which the limit exists. There is no reason to expect Z to be a bounded operator and no reason to expect $\text{Dom}(Z)$ to exhaust \mathcal{B} . The first main result of this paper is the following theorem.

Theorem 1. *Suppose that $T_t : \mathcal{B} \rightarrow \mathcal{B}$ is a bounded one-parameter semigroup satisfying Assumptions 1, 2, and 3A. Then there exists a finite set*

$$\{z_j\}_{j=1}^N \subset \{z \in \mathbb{C} : -\lambda < \Re(z) \leq 0, |\Im(z)| \leq \beta\},$$

a set of finite rank projectors $\{\Pi_j\}_{j=1}^N \subset \mathcal{B}(\mathcal{B}, \mathcal{B})$ and an operator-valued function $t \mapsto P_t \in \mathcal{B}(\mathcal{B}, \mathcal{B})$ such that

$$T_t = P_t + \sum_{j=1}^N e^{tz_j} \Pi_j \quad \text{for all } t \geq 0. \quad (2.3)$$

Moreover for all $\ell < \lambda$ there exists $C_\ell > 0$ such that, for all $\mu \in \text{Dom}(Z)$, $t \geq 0$,

$$\|P_t \mu\|_{\mathcal{A}} \leq C_\ell e^{-\ell t} \|Z\mu\|_{\mathcal{B}}. \quad (2.4)$$

The proof of the theorem is the content of Section 4.

Remark 2.1. The theorem is only useful if the set $\text{Dom}(Z)$ is sufficiently large. However if T_t were a strongly-continuous one-parameter semigroups this is a closed and dense subset of \mathcal{B} . Standard approximations will typically allow the result to be useful.

Remark 2.2. With the current ideas we cannot hope for a strengthening of the theorem whereby $\|P_t\|_{\mathcal{B}} \leq C e^{-\ell t}$. This is a subtlety of one-parameter semigroups as demonstrated by Zabczyk's example³ [6, Theorem 8.2.9].

Remark 2.3. If the one-parameter semigroup was actually a one-parameter semigroup of operators associated to an ergodic flow, as in the intended applications, then one can typically show that mixing is equivalent to $\{z_j\}_{j=1}^N \cap \{\Re(z) = 0\} = \{0\}$ (see, for example [5]).

Remark 2.4. Most often Assumption 2 is proven by the combination of a compact embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ and an estimate of the form $\|R(z)^n \mu\|_{\mathcal{B}} \leq C(\Re(z) + \lambda)^{-n} \|\mu\|_{\mathcal{B}} + C \|\mu\|_{\mathcal{A}}$. Such information is sufficient to deduce the estimate of the essential spectral radius by following Hennion's argument [13] based on the formula by Nussbaum [21] (see for example [17]). In this case Assumption 3A can be weakened: It is then sufficient to prove the estimate of Assumption 3A in the weaker norm $\|\cdot\|_{\mathcal{A}}$ rather than in the original norm $\|\cdot\|_{\mathcal{B}}$ and only for $\mu \in \mathcal{B}$ for which $\|\mu\|_{\mathcal{B}}$ is sufficiently small in comparison to $\|\mu\|_{\mathcal{A}}$.

In order to state the result which corresponds to rapid mixing we must have higher order control on the regularity in the flow direction. For any $q \in \mathbb{N}$ define the norm

$$\|\mu\|_{Z^q} := \sum_{0 \leq n \leq q} \|Z^n \mu\|_{\mathcal{B}},$$

for all $\mu \in \text{Dom}(Z^q)$. The second main result of this paper is the following theorem.

Theorem 2. Suppose that $T_t : \mathcal{B} \rightarrow \mathcal{B}$ is a bounded one-parameter semigroup satisfying Assumptions 1, 2, and 3B. Then there exists a finite set

$$\{z_j\}_{j=1}^N \subset \{z \in \mathbb{C} : -\lambda < \Re(z) \leq 0, |\Im(z)| \leq \beta\},$$

a set of finite rank projectors $\{\Pi_j\}_{j=1}^N \subset \mathcal{B}(\mathcal{B}, \mathcal{B})$ and an operator-valued function $t \mapsto P_t \in \mathcal{B}(\mathcal{B}, \mathcal{B})$ such that

$$T_t = P_t + \sum_{j=1}^N e^{tz_j} \Pi_j \quad \text{for all } t \geq 0. \quad (2.5)$$

Moreover for all $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$, $C_p > 0$ such that, for all $\mu \in \text{Dom}(Z^q)$, $t \geq 0$,

$$\|P_t \mu\|_{\mathcal{A}} \leq C_p t^{-p} \|\mu\|_{Z^q}. \quad (2.6)$$

The proof of the theorem is the content of Section 5.

Remark 2.5. Note that the required regularity $q = q(p)$ depends on the desired decay rate p and must be taken larger when p increases. The exact connection of the two can be seen in the calculation at the end of Section 5. When considering rates of mixing of a flow the above requirement of $\mu \in \text{Dom}(Z^q)$ becomes the unfortunate requirement of the observables being “rather smooth” in the flow direction.

³There exists a one-parameter group T_t acting on a Hilbert space such that the spectrum of Z is contained in $i\mathbb{R}$ but e^{itZ} is in the spectrum of T_t for all $t \in \mathbb{R}$.

Remark 2.6. Dolgopyat's original formulation [9] of rapid mixing considered \mathcal{C}^∞ functions as observables. See [18, Definition 2.2] for a formulation closer to the above statement. Note that being *sufficiently regular* in the flow direction is crucial for this result. However it is of some help that the notion of regularity is entirely dependent on the choice of $\|\cdot\|_{\mathcal{B}}$.

Remark 2.7. Usually Assumption 3B is proven by showing the non-existence of *approximate eigenvalues* [9, 10, 18, 11, 19].

3. APPLICATIONS

Assumptions 1, 2, and Assumption 3A have been shown for contact Anosov flows [17] apart from Assumption 1 (in the reference the two spaces \mathcal{B} and \mathcal{A} are denoted $\mathcal{B}(\mathcal{M}, \mathbb{C})$ and $\mathcal{B}_w(\mathcal{M}, \mathbb{C})$ respectively). It is convenient to modify the stronger of the two norms by adding a term which controls (in supremum) the derivative in the flow direction. As a result assumption 1 is simple to prove in this setting once one notices that $\int_0^t V\eta \circ \Phi^s ds = \eta \circ \Phi^t - \eta$ for all $t \geq 0$ where V is the vector field associated to the flow $\Phi^t : \mathcal{M} \rightarrow \mathcal{M}$. Let \mathcal{L}_t be the associated transfer operator. This means that

$$\int_{\mathcal{M}} (\mathcal{L}_t h - h) \cdot \eta dm = \int_{\mathcal{M}} h \cdot (\eta \circ \Phi^t - \eta) dm = \int_0^t \int_{\mathcal{M}} \mathcal{L}_s h \cdot V\eta dm ds.$$

This immediately implies the weak Lipschitz control required by Assumption 1. Similarly these assumptions have been shown to be satisfied in several other settings [2, 12].

Assumptions 1, 2, and Assumption 3B have been shown for a prevalent set of Axiom A flows in [9]. However in the reference everything is described in the *twisted transfer operator* language for suspension flows. To pass from that viewpoint to the present language note that the calculation (see for example [22] or [1, Lemma 7.17]) used to relate the Laplace transform of the correlation to a sum of twisted transfer operators may equally well be used for the Laplace transform of the transfer operator of the flow for the suspension flow.

4. THE EXPONENTIALLY MIXING CASE

Throughout we suppose that Assumptions 1, 2, and 3A are satisfied. First we recall a fact which appeared in [3].

Lemma 4.1 ([3, Lemma 2.2]). *For all $\Re(z) > 0$, $\Re(\zeta) > 0$ then, on $\mathcal{B}(\mathcal{B}, \mathcal{B})$, holds*

$$(z - \zeta)R(\zeta)R(z) = R(\zeta) - R(z).$$

We already know that the operator valued function $z \mapsto R(z) \in \mathcal{B}(\mathcal{B}, \mathcal{B})$ is holomorphic on the set $\{z \in \mathbb{C} : \Re(z) > 0\}$. We now take advantage of Assumption 3A for the following result.

Lemma 4.2. *The operator valued function $z \mapsto R(z) \in \mathcal{B}(\mathcal{B}, \mathcal{B})$ admits an extension which is meromorphic on the set $\{z \in \mathbb{C} : \Re(z) > -\lambda\}$ and holomorphic on the set $\{z \in \mathbb{C} : \Re(z) > -\lambda, |\Im(z)| \geq \beta\}$.*

Proof. Consider $z \in \mathbb{C}$, $\Re(z) > 0$ and $\eta \in \mathbb{C}$, $|\eta| > \Re(z)^{-1}$. By Lemma 4.1 $\eta^{-1}R(z + \eta^{-1})R(z) = R(z + \eta^{-1}) - R(z)$ since in particular $\eta \neq 0$ and $\Re(z - \frac{1}{\eta}) > 0$. Consequently

$$R(z + \frac{1}{\eta}) = \eta R(z)(\eta \text{id} - R(z))^{-1}. \quad (4.1)$$

We know that $(\eta \text{id} - R(z))$ is invertible since the spectral radius of $R(z)$ is not greater than $\Re(z)^{-1}$. Consequently (4.1) defines the extension of $R(z)$ into the left half of the imaginary plane. By Assumption 2 the operator valued function $\eta \mapsto$

$(\eta \text{id} - R(z))^{-1}$ is meromorphic on the set $\{|\eta| > (\Re(z) + \lambda)^{-1}\}$. By Assumption 3A we know that the spectral radius of $R(z)$ is not greater than $(\Re(z) + \lambda)^{-1}$ when $\Re(z) > -\lambda$ and $|\Im(z)| \geq \beta$. This means that in this case the operator valued function $\eta \mapsto (\eta \text{id} - R(z))^{-1}$ is holomorphic on this set. \square

Proof of the first part of Theorem 1. An immediate consequence of Lemma 4.2 is that the function $z \mapsto R(z) \in \mathcal{B}(\mathcal{B}, \mathcal{B})$ has no more than a finite number of poles on the set $\{z \in \mathbb{C} : \Re(z) > -\lambda\}$. We let $\{z_j\}_{j=0}^N \subset \mathbb{C}$ denote this finite set of poles. For each z_j let

$$\Pi_j := \frac{1}{2\pi i} \int_{\Gamma_j} R(z) dz$$

where Γ_j is a positively-orientated small circle enclosing z_j but excluding all other singularities of $R(z)$. As is well known for spectral projectors, the resolvent equation, which was proven in Lemma 4.1, implies that the definition is independent on the choice of Γ_j subject to the above conditions. We now, for all $t \geq 0$, define $P_t : \mathcal{B} \rightarrow \mathcal{B}$ by

$$P_t := T_t - \sum_{j=1}^N e^{tz_j} \Pi_j.$$

To complete the proof of the theorem it remains to give the appropriate estimates on P_t . This is the substantial part of the present argument and will be postponed until the end of the section. \square

We need one more fact which appeared in [3]. It is an application of the inverse of the Laplace-Stieltjes transform of an operator valued function to the present situation.

Lemma 4.3 ([3, Theorem 1]). *Suppose $t \geq 0$, $a > 0$. Then, in $\mathcal{B}(\mathcal{B}, \mathcal{A})$, we have that*

$$T_t = \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \int_{-k}^k e^{(a+ib)t} R(a+ib) db.$$

The whole idea of the present argument is to obtain better information on $R(z)$ and then use the formula given by the above lemma whilst shifting the contour.

Lemma 4.4. *Suppose that $\ell \in (0, \lambda)$. For all $b \in \mathbb{R}$, $|b| \geq \beta$, on $\mathcal{B}(\mathcal{B}, \mathcal{B})$*

$$R(-\ell + ib) = R(\alpha + ib) \left(\sum_{n=0}^{\infty} (\alpha + \ell)^n R(\alpha + ib)^n \right).$$

Moreover there exists $C_5 > 0$ such that for all $|b| \geq \beta$

$$\left\| \sum_{n=0}^{\infty} (\alpha + \ell)^n R(\alpha + ib)^n \right\|_{\mathcal{B}} \leq C_5 |b|^{\gamma_0}$$

where $\gamma_0 := \gamma \ln(1 + \ell \alpha^{-1}) \in (0, 1)$.

Proof. Since the extension of $R(z)$ was defined in Lemma 4.2 by the resolvent equation we have

$$\begin{aligned} R(-\ell + ib) &= R(\alpha + ib) [\text{id} - (\alpha + \ell)R(\alpha + ib)]^{-1} \\ &= R(\alpha + ib) \left(\sum_{n=0}^{\infty} (\alpha + \ell)^n R(\alpha + ib)^n \right). \end{aligned}$$

It is convenient to split the sum as

$$\sum_{n=0}^{\infty} (\alpha + \ell)^n R(\alpha + ib)^n = \sum_{k=0}^{\infty} (\alpha + \ell)^{k\tilde{n}(b)} R(\alpha + ib)^{k\tilde{n}} \sum_{m=0}^{\tilde{n}(b)-1} (\alpha + \ell)^m R(\alpha + ib)^m,$$

where $\tilde{n}(b) = \lceil \gamma \ln |b| \rceil$. We use the estimate $\|R(\alpha + ib)\|_{\mathcal{B}} \leq C_1 \alpha^{-1}$ and the estimate $\|R(\alpha + ib)^{\tilde{n}(b)}\|_{\mathcal{B}} \leq C_3(\alpha + \lambda)^{-\tilde{n}(b)}$ of Assumption 3A. The norm of the first sum decreases as $|b|$ increases and so we have

$$\sum_{k=0}^{\infty} (\alpha + \ell)^{k\tilde{n}(b)} \|R(\alpha + ib)^{k\tilde{n}(b)}\|_{\mathcal{B}} \leq C_6,$$

where $C_6 := C_3[1 - (\frac{\alpha+\ell}{\alpha+\lambda})^{\gamma \ln \beta}]^{-1}$. The norm of the second sum is increasing as $|b|$ increases. We have

$$\begin{aligned} \sum_{m=0}^{\tilde{n}(b)-1} (\alpha + \ell)^m \|R(\alpha + ib)^m\|_{\mathcal{B}} &\leq C_1 \sum_{m=0}^{\tilde{n}(b)-1} \left(\frac{\alpha+\ell}{\alpha}\right)^m \\ &\leq C_1 \alpha \ell^{-1} |b|^{\gamma_0}, \end{aligned}$$

recalling that $\gamma_0 = \gamma \ln(1 + \ell \alpha^{-1})$. We let $C_5 := C_1 C_6 \alpha \ell^{-1}$. The above two estimates complete the proof of the lemma. \square

Lemma 4.5. *There exists $C_7 > 0$ such that for all $|b| \geq \beta$*

$$\|R(\alpha + ib)\|_{\mathcal{B} \rightarrow \mathcal{A}} \leq C_7 |b|^{-1}.$$

Proof. This lemma is a consequence of Assumption 1. Fix $b \in \mathbb{R}$. For all $n \in \mathbb{N}$ let $t_n := 2\pi n |b|^{-1}$ and hence

$$\begin{aligned} R(\alpha + ib) &= \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} e^{-(\alpha+ib)t} T_t dt \\ &= \sum_{n=0}^{\infty} e^{-\alpha t_n} \int_{t_n}^{t_{n+1}} e^{-ibt} \left(e^{-\alpha(t-t_n)} T_t - T_{t_n} \right) dt, \end{aligned}$$

since $\int_{t_n}^{t_{n+1}} e^{-ibt} dt = 0$. We have that $|e^{-\alpha(t-t_n)} - 1| \leq \alpha(t-t_n) \leq 2\pi\alpha |b|^{-1}$. Using Assumption 1 we have that $\|T_t - T_{t_n}\|_{\mathcal{B} \rightarrow \mathcal{A}} \leq (t-t_n)C_2C_1 \leq 2\pi C_2C_1 |b|^{-1}$ for all $t \in (t_n, t_{n+1})$. This means that

$$\left\| e^{-\alpha(t-t_n)} T_t - T_{t_n} \right\|_{\mathcal{B} \rightarrow \mathcal{A}} \leq 2\pi C_1 (\alpha + C_2) |b|^{-1}.$$

On the other hand

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-\alpha t_n} \int_{t_n}^{t_{n+1}} dt &= \sum_{n=0}^{\infty} \frac{2\pi}{|b|} e^{-\alpha 2\pi n |b|^{-1}} \\ &= \frac{2\pi |b|^{-1}}{1 - e^{-\alpha 2\pi |b|^{-1}}} \leq \frac{2\pi \beta^{-1}}{1 - e^{-\alpha 2\pi \beta^{-1}}} =: C_8. \end{aligned}$$

We have shown that $\|R(\alpha + ib)\|_{\mathcal{B} \rightarrow \mathcal{A}} \leq C_7 |b|^{-1}$ where $C_7 := 2\pi C_1 C_8 (\alpha + C_2)$. \square

It is known [6, Lemma 6.1.15] that $\text{Dom}(Z)$ is complete with respect to the norm

$$\|\mu\|_Z := \|Z\mu\|_{\mathcal{B}} + \|\mu\|_{\mathcal{B}}.$$

From this point forward when we refer to $\text{Dom}(Z)$ this should be understood to imply the Banach space $(\text{Dom}(Z), \|\cdot\|_Z)$.

Lemma 4.6. *For all $z \in \mathbb{C}$ in the holomorphic domain of $R(z)$ and $z \neq 0$, on $\mathcal{B}(\text{Dom}(Z), \mathcal{B})$*

$$R(z) - \frac{1}{z} \text{id} = \frac{1}{z} R(z) Z.$$

Proof. By standard [6, Theorem 8.2.1] semigroup theory $R(z) = (z \text{id} - Z)^{-1}$. This means that $R(z)(z \text{id} - Z) = \text{id} = z R(z) - R(z) Z$. \square

Proof of the second part of Theorem 1. Let $\Re(a) > 0$ and let $\ell < \lambda$ such that $\Re(z_j) > -\ell$ for all j . By Lemma 4.3 and shifting the contour of integration, remembering that $R(z)$ has a pole at each $\{z_j\}_{j=1}^N$, we have, on $\mathcal{B}(\mathcal{A}, \mathcal{B})$, for all $t \geq 0$

$$\begin{aligned} T_t &= \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \int_{-k}^k e^{(a+ib)t} R(a+ib) db \\ &= \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \int_{-k}^k e^{(-\ell+ib)t} R(-\ell+ib) db + \sum_{j=1}^N \frac{e^{tz_j}}{2\pi i} \int_{\Gamma} R(z) dz. \end{aligned}$$

This means that, for all $t \geq 0$,

$$P_t = \lim_{k \rightarrow \infty} \frac{e^{-\ell t}}{2\pi i} \int_{-k}^k e^{ibt} R(-\ell+ib) db. \quad (4.2)$$

Since $\int_{-\infty}^{\infty} \frac{e^{(\ell+ib)t}}{\ell+ib} db = 0$ we have

$$P_t = \lim_{k \rightarrow \infty} \frac{e^{-\ell t}}{2\pi i} \int_{-k}^k e^{ibt} \left(R(-\ell+ib) + \frac{\mathbf{id}}{-\ell+ib} \right) db.$$

By Lemma 4.6 we have that, on \mathcal{A} , for every $\mu \in \text{Dom}(Z)$

$$P_t \mu = \lim_{k \rightarrow \infty} \frac{e^{-\ell t}}{2\pi i} \int_{-k}^k e^{ibt} \frac{R(-\ell+ib)Z\mu}{-\ell+ib} db.$$

We must estimate $\|P_t \mu\|_{\mathcal{A}}$. Note that $\|R(-\ell+ib)Z\mu\|_{\mathcal{A}} \leq \|R(-\ell+ib)\|_{\mathcal{B} \rightarrow \mathcal{A}} \|Z\mu\|_{\mathcal{B}}$. Let

$$C_9 := \frac{1}{2\pi} \int_{-\beta}^{\beta} \frac{\|R(-\ell+ib)\|_{\mathcal{B} \rightarrow \mathcal{A}}}{|-\ell+ib|} db.$$

Since the contour $\{z \in \mathbb{C}, \Re(z) = -\ell, |\Im(z)| \leq \beta\}$ was chosen to avoid all the singularities of $R(z)$ we have that $C_9 < \infty$. By Lemma 4.5 and Lemma 4.4 $\|R(-\ell+ib)\|_{\mathcal{B} \rightarrow \mathcal{A}} \leq C_7 C_5 |b|^{-(1-\gamma_0)}$ for all $|b| \geq \beta$. Since $(1-\gamma_0) \in (0, 1)$

$$C_{10} := \frac{1}{2\pi} \int_{\beta}^{\infty} |b|^{-(2-\gamma_0)} db < \infty.$$

We have shown that $\|P_t \mu\|_{\mathcal{A}} \leq C_{\ell} e^{-\ell t} \|Z\mu\|_{\mathcal{B}}$ where $C_{\ell} := (C_9 + 2C_{10}C_7C_5)$. \square

5. THE RAPID MIXING CASE

Throughout we suppose that Assumptions 1, 2, and 3B are satisfied.

Proof of the first part of Theorem 2. As before we use Lemma 4.3 to write that

$$T_t = \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \int_{-k}^k e^{(a+ib)t} R(a+ib) db.$$

Identically to the proof of the first part of Theorem 1 we deal with the part of the integral from $-\beta$ to β by selecting a finite set of projectors $\{\Pi_j\}_j$ corresponding to the poles $\{z_j\}_j$ of $R(z)$ in the region $\{z \in \mathbb{C} : \Re(z) > -\ell, |\Im(z)| \leq \beta\}$. We define (as before)

$$P_t := T_t - \sum_{j=1}^N e^{tz_j} \Pi_j.$$

It now remains to estimate $\|P_t \mu\|_{\mathcal{A}}$ in terms of $\|\mu\|_{Z^q}$ (for some $q \in \mathbb{N}$) crucially using Assumption 3B. It is convenient to shift the contour to $\{ib - \min(\epsilon, |b|^{-r}), b \in \mathbb{R}\}$ where $\epsilon \in (0, \ell)$ is chosen such that the new contour avoids all the singularities of $R(z)$. The central part of this integral gives an exponentially bounded term as

per (4.2) with a constant which depends on $\sup_{|b| \leq \beta} \|R(-\epsilon, b)\|_{\mathcal{B}} < \infty$. This means that we merely need to estimate the norm of

$$\lim_{k \rightarrow \infty} \int_{\beta}^k \exp(-t|b|^{-r}) e^{ibt} R(ib - |b|^{-r}) \mu \, db, \quad (5.1)$$

and the similar integral from $-k$ to $-\beta$. This will be postponed until the end of this section. \square

Now we will need the following higher order version of Lemma 4.6.

Lemma 5.1. *Let $n \in \mathbb{N}$. For all $z \in \mathbb{C}$ in the holomorphic domain of $R(z)$ and $z \neq 0$, on $\mathcal{B}(\text{Dom}(Z^n), \mathcal{B})$*

$$R(z) = \frac{1}{z^n} R(z) Z^n + \sum_{j=0}^{n-1} \frac{1}{z^{j+1}} Z^j.$$

Proof. The case $n = 1$ is Lemma 4.6. I.e., $R(z) = \frac{1}{z} R(z) Z + \frac{1}{z} \text{id}$. Simply iterating this formula proves the result for all $n \in \mathbb{N}$. \square

For the following it is essential that Assumption 3B is satisfied.

Lemma 5.2. *Let $n \in \mathbb{N}$. There exists $C_{11} > 0$ such that*

$$\|R(ib - |b|^{-r}) \mu\|_{\mathcal{B}} \leq C_{11} |b|^{s-n} \|\mu\|_{Z^n}$$

for all $\mu \in \text{Dom}(Z^n)$, $b \in \mathbb{R}$, $|b| \geq \beta$.

Proof. Using Lemma 5.1 we have

$$\|R(z) \mu\| \leq \frac{1}{|z|^n} \|R(z)\|_{\mathcal{B}} \|\mu\|_{Z^n} + \sum_{j=0}^{n-1} \frac{1}{|z|^{j+1}} \|\mu\|_{Z^j}.$$

We now substitute $z = ib - |b|^{-r}$. Since $\|R(z)\|_{\mathcal{B}} \leq C_4 |\Im(z)|^s$ by Assumption 3B there exists some $C_{11} > 0$ such that the lemma holds. \square

Proof of the second part of Theorem 2. We now use the above lemma to estimate the norm of the integral of (5.1) and so complete the proof of Theorem 2.

$$\begin{aligned} \lim_{k \rightarrow \infty} \left\| \int_{\beta}^k \exp(-t|b|^{-r}) e^{ibt} R(ib - |b|^{-r}) \mu \, db \right\|_{\mathcal{B}} \\ \leq C_{11} \left(\int_{\beta}^{\infty} \exp(-t|b|^{-r}) |b|^{s-q} \, db \right) \|\mu\|_{Z^q}. \end{aligned}$$

This holds for any $q \in \mathbb{N}$ but for our purposes we must choose q large, in particular larger than s . Estimating the integral⁴ and choosing q even larger depending also on the required rate of polynomial decay (denoted p in the statement of the theorem) concludes the estimate. \square

⁴Suppose that $a > 0$, $n, k \in \mathbb{N}$. Let $I(n) := \int_0^a e^{-tx} x^n \, dx$. Integrating by parts $I(n) \leq \frac{n}{t} I(n-1)$, and $I(0) \leq \frac{1}{t}$. Consequently $I(n) \leq n! t^{-n}$. By a change of variables $x = y^{-k}$ the integral $I(n)$ is equal to $k \int_{a^k}^{\infty} \exp(-ty^{-k}) y^{-(nk+k+1)} \, dy$.

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